Lecture 11.
Stability of Numerical Integration Methods

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Outline

- **Absolute stability (A.S.)**
- **Convergence problem in transient simulation**
- **Numerical stability of three methods**
- **Region of A.S. for LMS methods**

Absolute Stability

Three Integration Formulas

FE

$$y_n - y_{n-1} - h\,\dot{y}_{n-1} = 0$$

BE

$$y_n - y_{n-1} - h\,\dot{y}_n = 0$$

TR

$$y_n - y_{n-1} - \frac{h}{2}\left(\dot{y}_n + \dot{y}_{n-1}\right) = 0$$

LMS

$$\sum_{i=0}^{k} \alpha_i y_{n-i} + h \sum_{j=0}^{m} \beta_j \dot{y}_{n-j} = 0$$

All are iteration formulas.

The choice of "h" affects the convergence.

Different methods have different convergence properties.

Absolute Stability

- **“Absolute stability” considers how the choice of step-size (h) affects the convergence of an integration method.**

- **Characterized by a convergence region in the complex plane.**

- **The convergence region is found by a simple test model.**

A Simple Test Model

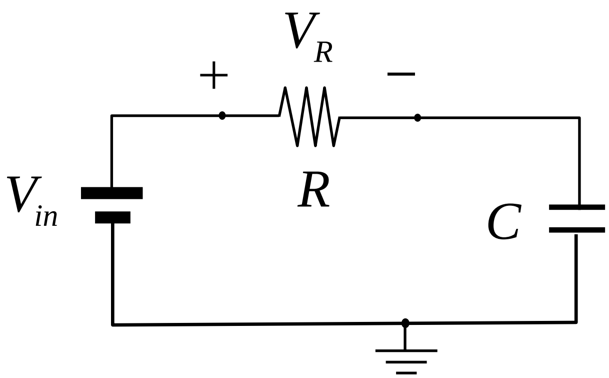
- **Use a scalar model to test how local errors are accumulated:**

Test model $$\frac{dx(t)}{dt} = -x(t)$$

The exact solution is:

$$x(t) = e^{-t}$$

Initial condition: $x(0) = 1$



Find the voltage across R:

$$V_C(0) = 0$$

$$C\frac{d(V_{in} - V_R)}{dt} = \frac{V_R}{R}$$

$$V_R(0) = V_{in}$$

$$\frac{dV_R}{dt} = -\frac{V_R}{RC} \Rightarrow V_R = V_{in}e^{-\frac{t}{RC}}$$

Why Choose a 1D Test Problem?

- **General nonlinear model (n-dimensional)**
  - **$dx/dt = F(x)$; $x \in R^n$;**
- **Linearization:**
  - **$dx/dt = Ax$, $A = \partial F(x_0) / \partial x$ (Jacobian)**
- **Diagonalization:**
  - **$\exists P$, $P^{-1}AP = \Lambda$ if all $\lambda_i(A)$ are distinct;**
  - **$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$**
  - $d\xi/dt = \Lambda\, \xi$, $x = P\xi$ **(state transform)**
  - $d\xi_i /dt = \lambda_i\, \xi_i$, $i = 1, \ldots, n$ (scalar models)

Test Problem

- **All n-dimensional non-linear models can be characterized locally by scalar models:**

$$\dot{x} = \lambda x: \qquad x(0) = 1; \qquad x \in \mathbb{R}$$

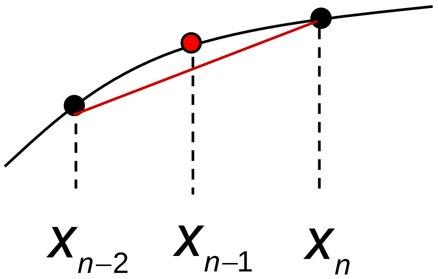
↑

$$\lambda \in \mathbb{C} \quad \text{a complex number}$$

Test a numerical method

Suppose we use a method called "Explicit Mid-Point (EMP)" for numerical integration;

$$\dot{x}_{n-1} = \frac{x_n - x_{n-2}}{2h} \quad \Rightarrow \quad x_n = x_{n-2} + 2h\,\dot{x}_{n-1}$$



Use this formula to solve the following test problem:

$$\dot{x} = -x, \quad x(0) = 1$$

Local Error Accumulation

$$\dot{x}_{n-1} = \frac{x_n - x_{n-2}}{2h} \quad \Rightarrow \quad x_n = x_{n-2} + 2h\,\dot{x}_{n-1}$$

$$\dot{x} = -x, \quad x(0) = 1$$

- **Exact solution known:**

$$x(t) = e^{-t}$$

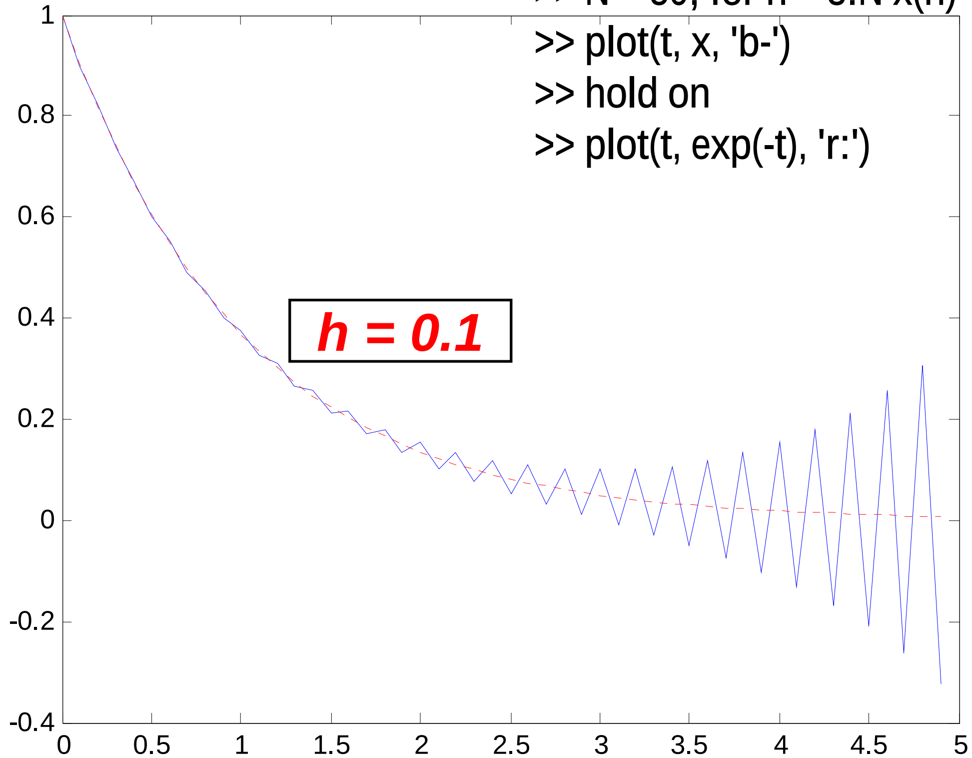
- **Choose h = 0.1:** $x_n = x_{n-2} - 2h\, x_{n-1}$
- $x_1 = x_0 + hx'_0$ **(Use Forward Euler for the 1st step)**

$$x_0 = 1,\ x_{0.1} = .9,\ x_{0.2} = .82,\ x_{0.3} = .736,\ \ldots,\ x_{9.9} = 44.0273186,\ x_{10} = -48.6495411$$

Diverges ...

MATLAB Simulation

```
>> h = 0.1;
>> t = [0, h];  x = [1, 1-h];
>> N = 50; for n = 3:N x(n) = x(n-2) - 2*h*x(n-1); t(n) = t(n-1) + h; end
>> plot(t, x, 'b-')
>> hold on
>> plot(t, exp(-t), 'r:')
```



Good accuracy at the beginning;

but diverges finally.

What caused the problem?

What if choosing a smaller step ?

$$x_n = x_{n-2} + 2h\dot{x}_{n-1}$$

$$x_1 = x_0 + h\dot{x}_0 = (1-h)x_0 \qquad \text{(for the 1st step)}$$

Choose h = 0.01:

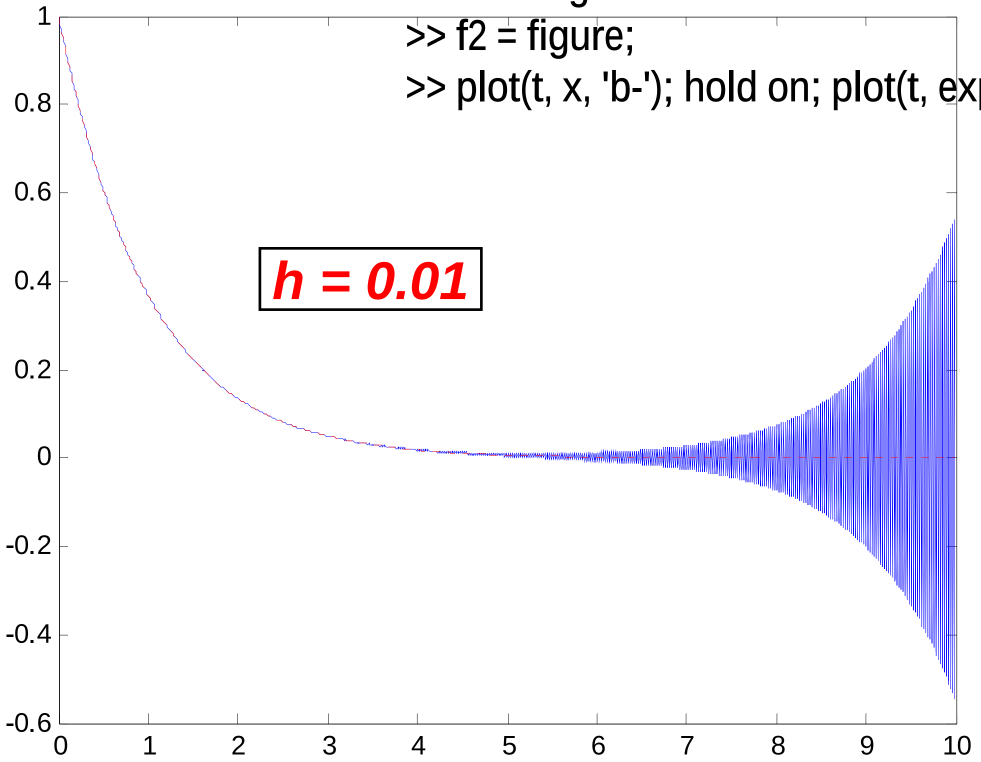
Still diverges (why?)

$$x_0 = 1, x_{.01} = .99, \ldots, x_{.1} = .3679, \ldots, x_1 = .55, \ldots, x_{12} = 12124.17839$$

Will a smaller "h" make it stable? --- actually not !!

MATLAB Simulation

```
>> h = 0.01;
>> t = [0, h];  x = [1, 1-h];
>> N = 1000; for n = 3:N x(n) = x(n-2) - 2*h*x(n-1); t(n) = t(n-1) + h; end
>> clear figure
>> f2 = figure;
>> plot(t, x, 'b-'); hold on; plot(t, exp(-t), 'r:');
```



Good accuracy at the beginning;

still diverges eventually.

The Reason ?

Loot at the iteration:

$$x_n = x_{n-2} - 2h \cdot x_{n-1}, \qquad (h > 0)$$

Suppose $x_n = c\,\lambda^n$ is a solution.

Substitute into the iteration: $c\lambda^n = c\lambda^{n-2} - 2h \cdot c\lambda^{n-1}$

$\Rightarrow$ $\lambda^2 + 2h\lambda - 1 = 0$

the characteristic equation

Find the too roots (characteristic values):

$$\lambda_1 = -h + \sqrt{h^2 + 1}, \qquad \lambda_2 = -h - \sqrt{h^2 + 1} < -1$$

Check the Characteristic Roots

$$x_n = x_{n-2} - 2h \cdot x_{n-1},$$

$$\Downarrow$$

The general solution is:

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

where c_1 and c_2 are constants to be determined by initial conditions.

(unstable)

$$\lambda_1 = -h + \sqrt{h^2 + 1}, \qquad \lambda_2 = -h - \sqrt{h^2 + 1} < -1 \qquad (h > 0)$$

The two characteristic roots determine the **convergence of x_n** !

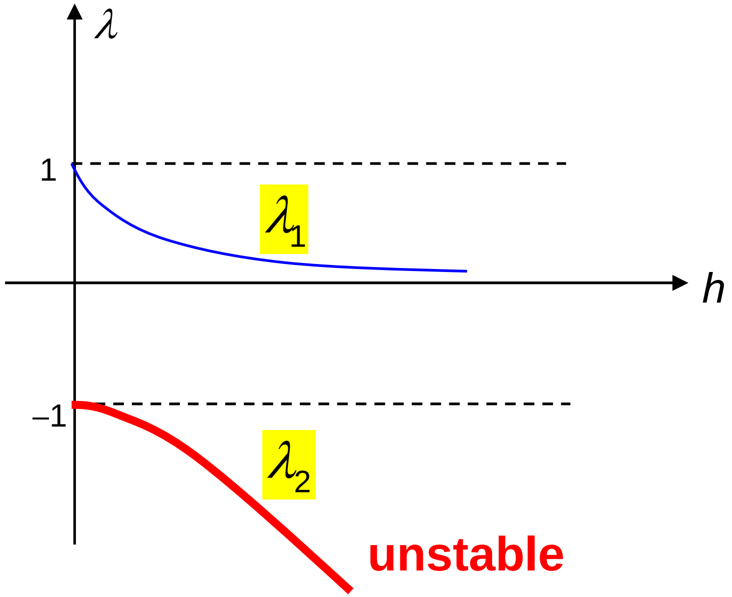
Plot the roots

$$\lambda_1 = -h + \sqrt{h^2 + 1},$$

$$\lambda_2 = -h - \sqrt{h^2 + 1} < -1$$

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

Unless the initial condition **makes $c_2 = 0$**, the iteration always diverges.



(cont'd)

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$

But if $c_2 = 0$, we'll get **h = 0** (which is not allowed.)

$$c_2 = 0 \Rightarrow x_n = c_1\lambda_1^n$$

The initial conditions are:

$x_0 = 1$ (given); $\boldsymbol{x_1 = (1-h)x_0 = 1-h}$ (by F. E.)

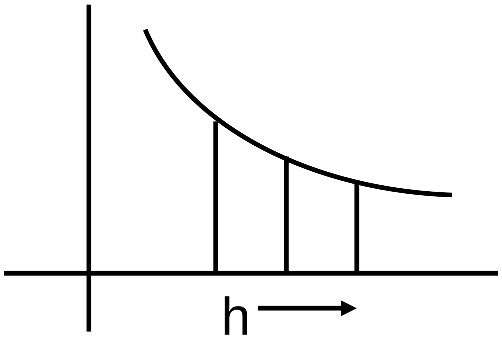
$$x_0 = 1 \Rightarrow c_1 = 1$$

$$x_1 = 1 - h \Rightarrow \lambda_1 = 1 - h$$

$$\left.\begin{array}{l} \lambda_1 = 1 - h \\ \lambda_1 = -h + \sqrt{h^2 + 1}, \end{array}\right\} \Rightarrow h = 0$$

Numerical Behavior

Example: $\dot{x} = -x$



- **Apply Forward Euler with h = 1:**

$$x_0 = 1, x_1 = 0, x_2 = 0, x_3 = 0$$

- **Apply Forward Euler with h = 3:**

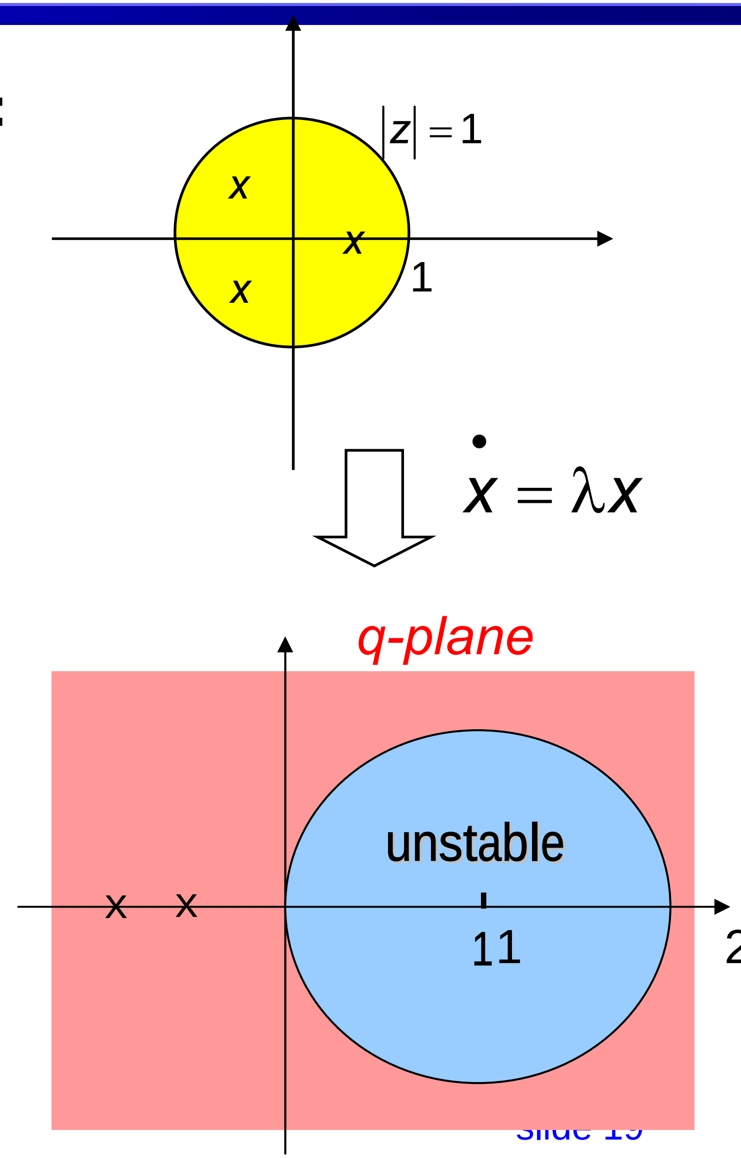
$$x_0 = 1, x_1 = -2, x_2 = 4, x_3 = -8, x_4 = 16, x_5 = -32$$

(diverges)

However, **Backward Euler and Trapezoidal Rule would not diverge.**

Stability Region

- **Use a simple test model x' = λx (λ is complex) to determine a region for the step-size *h***



- **Better if region is larger.**

- **Stability region can be derived algebraically.**

Characterization Method

1. **Choose an integration method with step size "h > 0".**
2. **Apply it to the test problem: $dx/dt = \lambda x$**
3. **Derive an algebraic characteristic equation.**
4. **Define a quantity: $q = \lambda h$ (as a complex number);**
5. **Find a region for q in the *C-plane* in which the integration method is stable.**

- **The region is called a "stability region".**

Absolute Stability

- **An integration method is "absolutely stable" if the stability region contains the point $q = 0$.**

Stability of Difference Equation

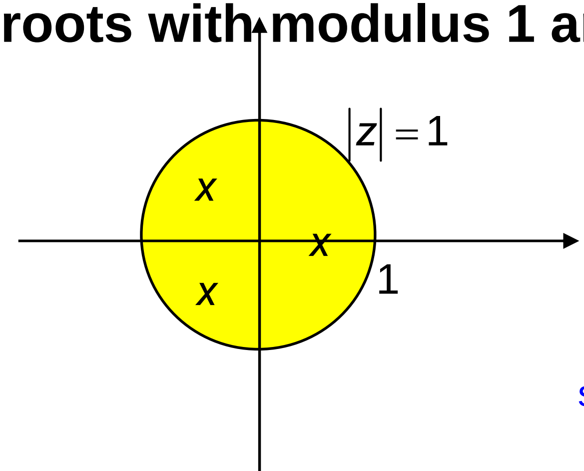
- **Theorem:** **The solutions of the difference equation**

$$\sum_{i=0}^{k} a_i x_{k-i} = 0$$

are bounded *if and only if* all roots of the *characteristic equation*

$$\sum_{i=0}^{k} a_i z^{k-i} = 0$$

$z_1, \ldots, z_r$ (r is the number of distinct roots) are inside or on the complex unit circle $\{ |z| \leq 1 \}$ and the roots with modulus 1 are of multiplicity 1.



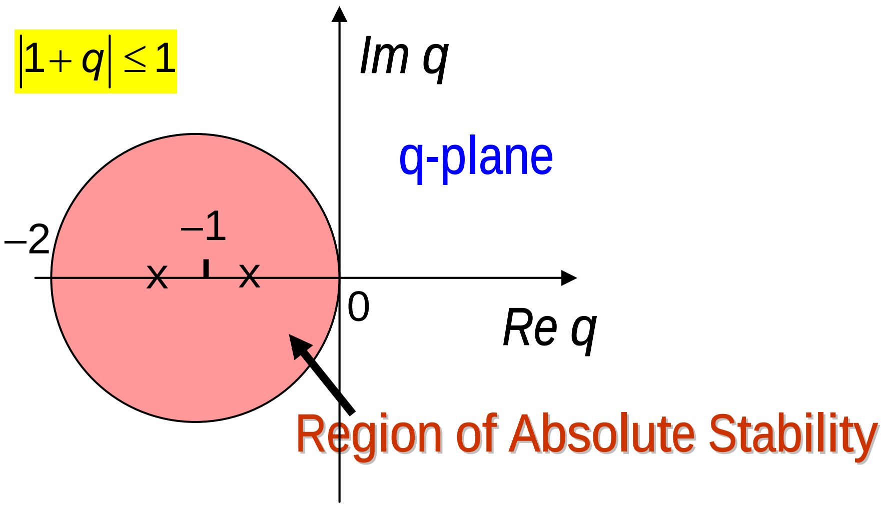
Forward Euler

$$x_n = x_{n-1} + h\dot{x}_{n-1}$$
$$= x_{n-1} + \lambda h x_{n-1}$$
$$= x_{n-1} + q x_{n-1}$$

$$\dot{x} = \lambda x$$

$$q = \lambda h$$

Char. eqn. $\quad z - (1+q) = 0 \quad \Rightarrow \quad |z| \le 1 \Leftrightarrow |1+q| \le 1$



Region of Absolute Stability

Numerical Stability:

Given $\lambda < 0$ (stable model), choose h small enough to have a stable method

Backward Euler

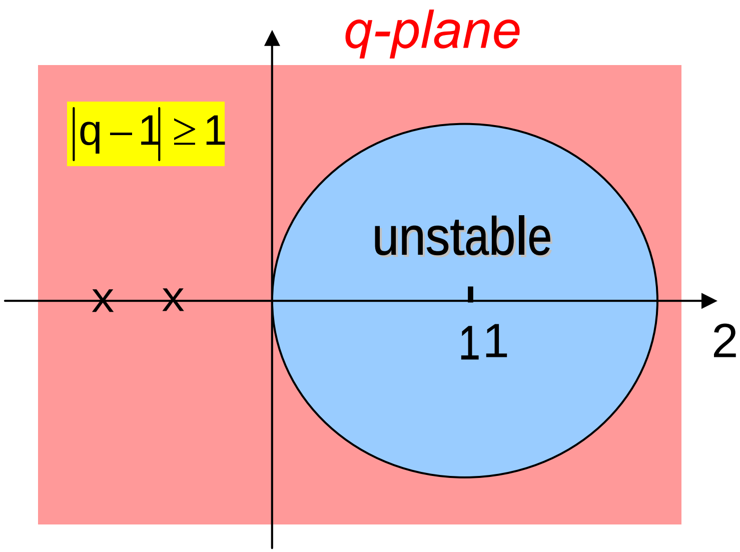
$$x_n = x_{n-1} + h\,\dot{x}_n$$

$$= x_{n-1} + \lambda h x_n$$

$$\boxed{q = \lambda h}$$

$$\dot{x} = \lambda x$$

⇨ $z(1-q) - 1 = 0$ ⇨ $|z| \le 1 \Leftrightarrow \left|\frac{1}{1-q}\right| \le 1$



Numerical Stability:

$q = \lambda h$ lies in the left-half plane for Re(λ) < 0 (stable model). Hence $|q-1| > 1$.

Thus, the method is <u>stable</u> for all h > 0 as long as the model is stable.

However, for Re(λ) > 0 (unstable model), the numerical solution may be <u>stable</u> for h large.

Trapezoidal Rule

$$x_n = x_{n-1} + \frac{h}{2}(\dot{x}_{n-1} + \dot{x}_n)$$

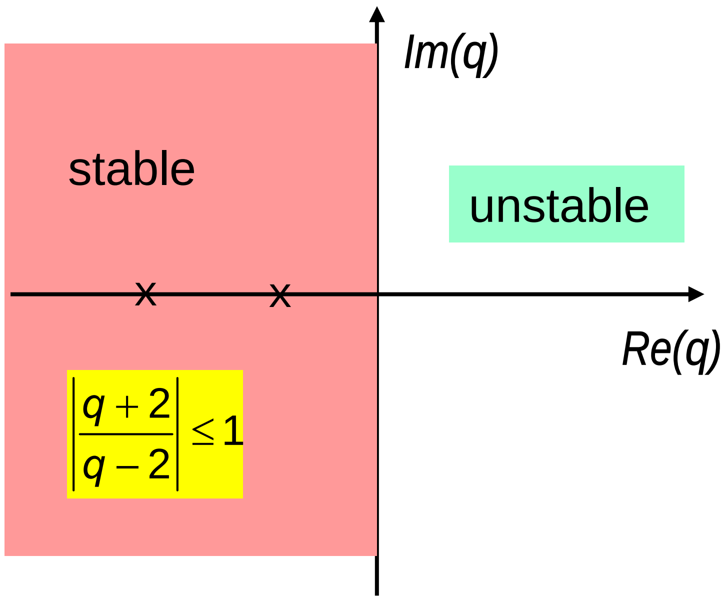
$$x_n = x_{n-1} + \frac{h\lambda}{2}(x_{n-1} + x_n)$$

$$\dot{x} = \lambda x$$

$$q = \lambda h$$

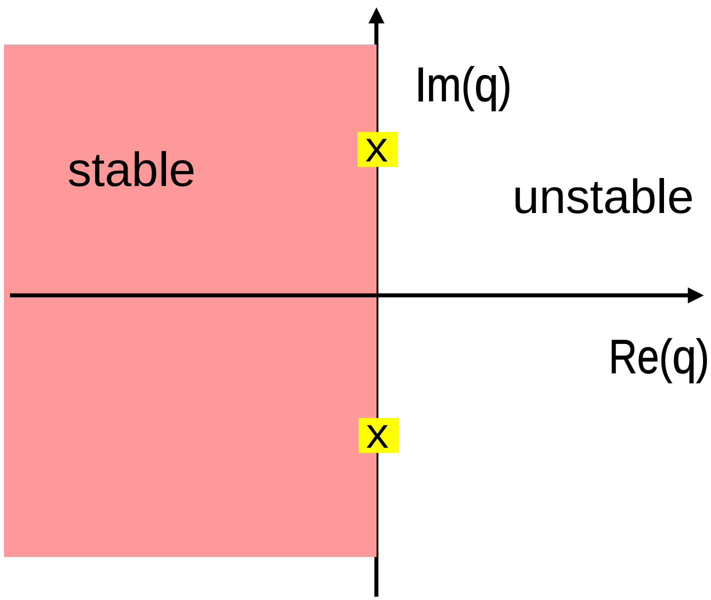
$$\left(1 - \frac{q}{2}\right)z - \left(1 + \frac{q}{2}\right) = 0$$

$$|z| \le 1 \Leftrightarrow \left|\frac{1 + q/2}{1 - q/2}\right| \le 1$$



TR is stable when the model is stable

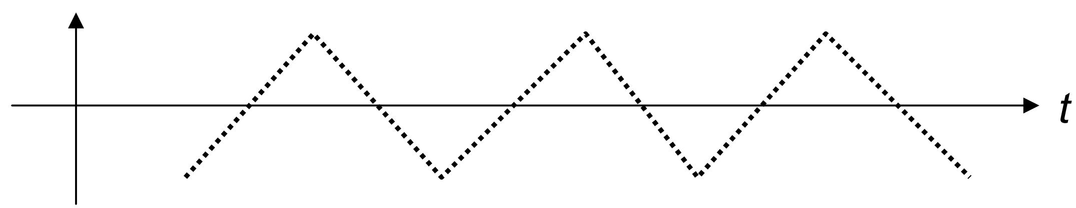
Trapezoidal Ringing



Problem:

If q= iα (pure imaginary), then the root is
$z = (1+i\alpha)/(1-i\alpha) \Rightarrow |z| = 1$.

We get "trapezoidal ringing."



Stability of LMS Methods

Consider a Linear Multi-Step method

$$\sum_{i=0}^{k} \alpha_i x_{n-i} + \sum_{i=0}^{k} \beta_i \dot{x}_{n-i} = 0$$

$$\dot{x} = \lambda x$$

⇒ $$\sum \alpha_i x_{n-i} + h\lambda \beta_i x_{n-i} = 0$$ **(difference equation)**

⇒ $$\sum_{i=0}^{k} (\alpha_i + q\beta_i) x_{n-i} = 0$$

let $q = \lambda h$

Difference Equation

- **Check the stability of this difference equation**

$$\sum_{i=0}^{k} (\alpha_i + q\beta_i) x_{n-i} = 0$$

$$x_n = c \cdot z^n$$

$$0 = c\left[(\alpha_0 + q\beta_0)z^n + (\alpha_1 + q\beta_1)z^{n-1} + \ldots + (\alpha_k + q\beta_k)z^{n-k}\right]$$

(char. eqn.)

$$(\alpha_0 + q\beta_0)z^k + (\alpha_1 + q\beta_1)z^{k-1} + \ldots + (\alpha_k + q\beta_k) = 0$$

Region of Absolute Stability

- The **region of absolute stability** of an LMS method is the set of $q = \lambda h$ (complex) such that all solutions of the difference equation

$$\sum_{i=0}^{k} (\alpha_i + q\beta_i) x_{n-i} = 0$$

remain bounded as n → ∞ .

- **A method is "absolutely stable" if the stability region contains the point q = 0.**

q = λh

Region of Absolute Stability

$$(1+q\beta_0)z^k + (\alpha_1 + q\beta_1)z^{k-1} + \ldots + (\alpha_k + q\beta_k) = 0$$

For what values of *q* do all the k roots of this polynomial lie in the unit disc { |z| ≤ 1 } ?

$$\left(z^k + \alpha_1 z^{k-1} + \ldots + \alpha_k\right) + q\left(\beta_0 z^k + \beta_1 z^{k-1} + \ldots + \beta_k\right) = 0$$

⇒

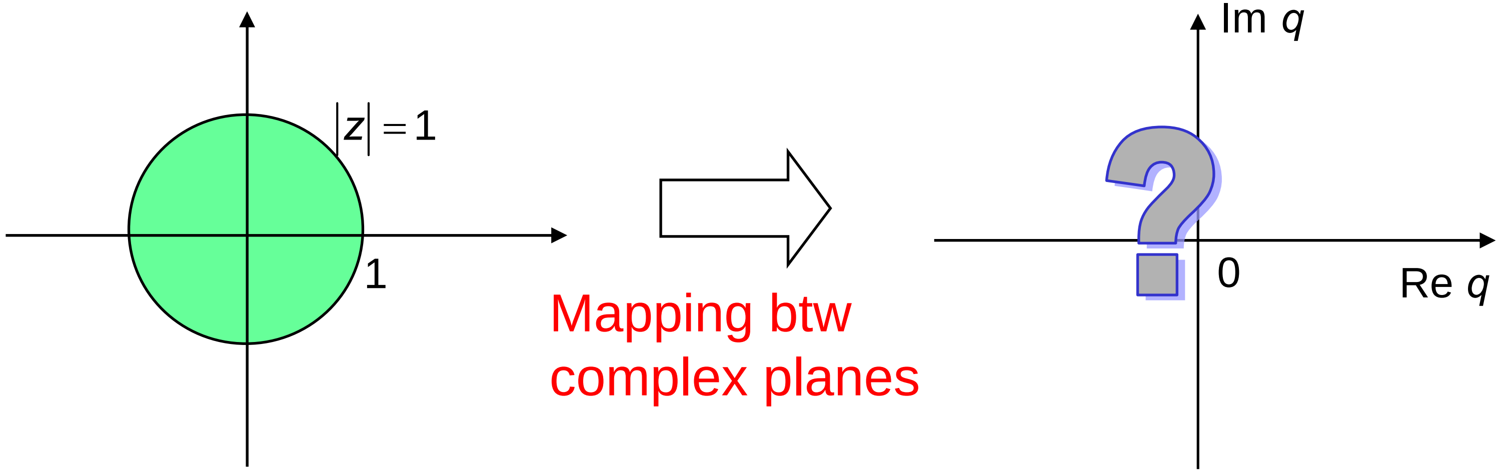
$$q = -\frac{p(z)}{\sigma(z)}$$

$$\begin{cases} p(z) = z^k + \alpha_1 z^{k-1} + \ldots + \alpha_k \\ \sigma(z) = \beta_0 z^k + \beta_1 z^{k-1} + \ldots + \beta_k \end{cases}$$

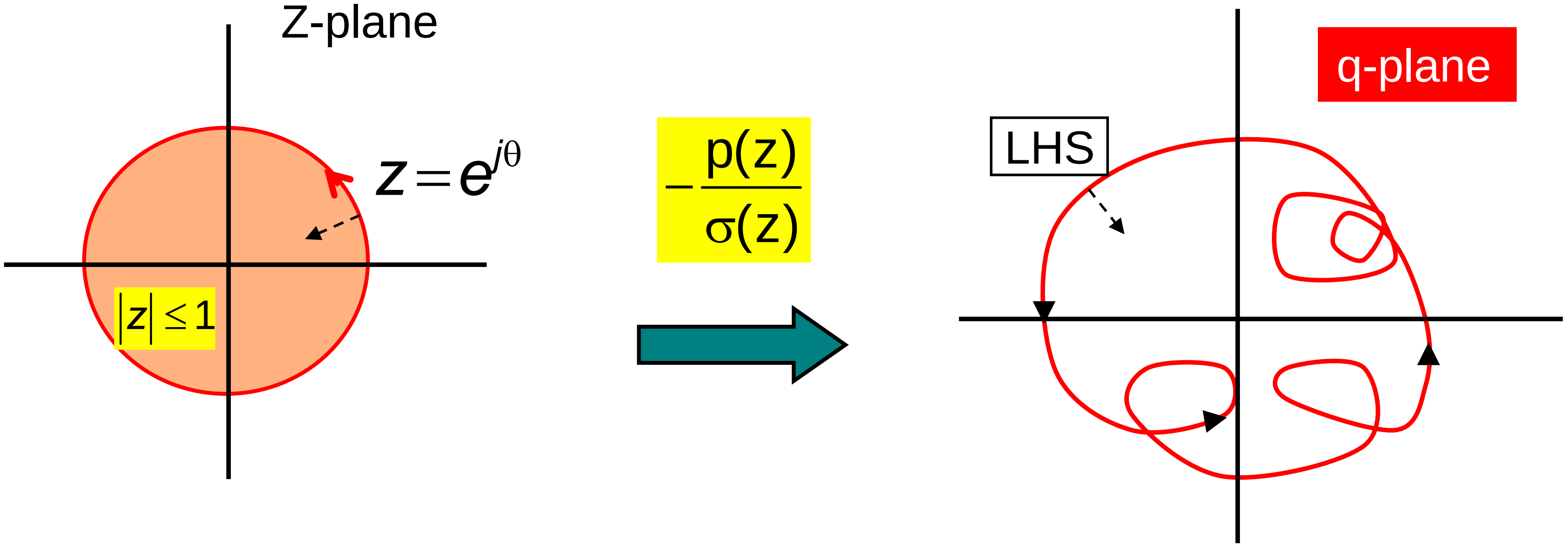
Region of Absolute Stability

The "region of absolute stability" is defined by the set

$$S \triangleq \{q \mid q = -p(z)/\sigma(z), \quad |z| \le 1\}$$



Conformal Mapping



Basic Results from Theory of Complex Variables

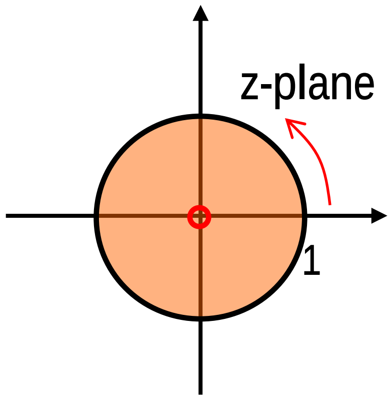
1. **Mapping –p(z) / σ(z) is conformal.**
2. **Region of "left-hand side" (LHS) to Region of LHS.**

Application to Mid-Point Method

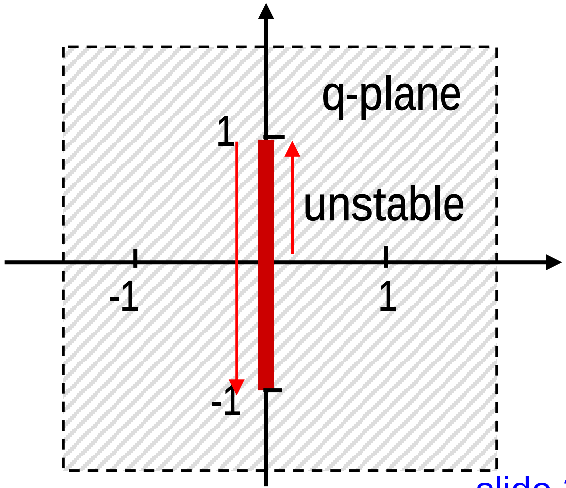
$$x_n = x_{n-2} + 2h\dot{x}_{n-1}$$

$$z^2 = 1 + 2qz \qquad z = e^{j\vartheta}$$

$$q = \frac{1}{2}\left(z - \frac{1}{z}\right) = \frac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right) = j\sin\theta$$



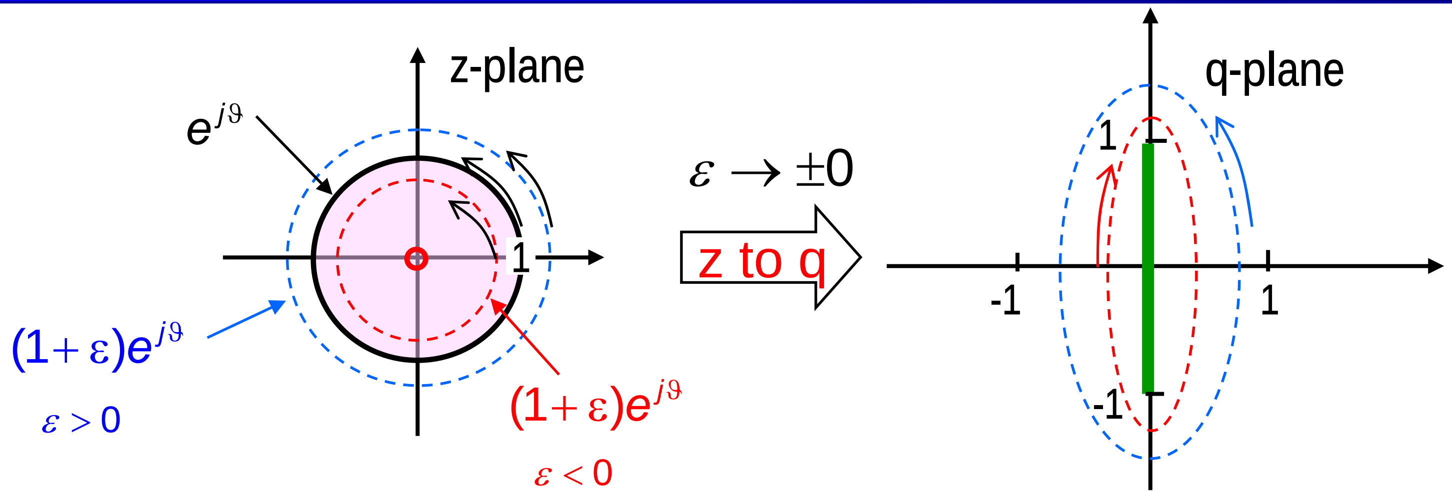
$$q = -\frac{p(z)}{\sigma(z)}$$



The stability region is just the interval [-j +j] on the jω axis.

Hence, the mid-point method is inherently unstable !

ε Analysis



$$q = \frac{1}{2}\left(z - \frac{1}{z}\right)$$

$$= \frac{1}{2}\left\{(1+\varepsilon)e^{j\theta} - \frac{1}{1+\varepsilon}e^{-j\theta}\right\}$$

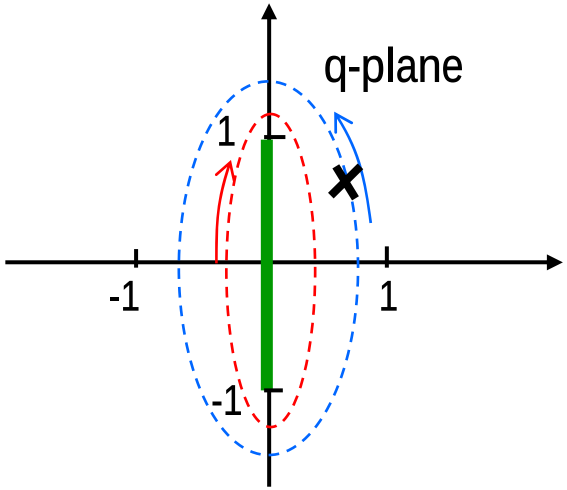
$$= \frac{1}{2}\left\{\left[(1+\varepsilon) - \frac{1}{1+\varepsilon}\right]\cos\theta + j\left[(1+\varepsilon) + \frac{1}{1+\varepsilon}\right]\sin\theta\right\}$$

$> 0; \quad \text{if } \varepsilon > 0$
$< 0; \quad \text{if } \varepsilon < 0$

[...] always > 1

The vertical line [-1, 1] is at the LHS of the blue ellipse and at the RHS of the red ellipse.

Interpretation - 1



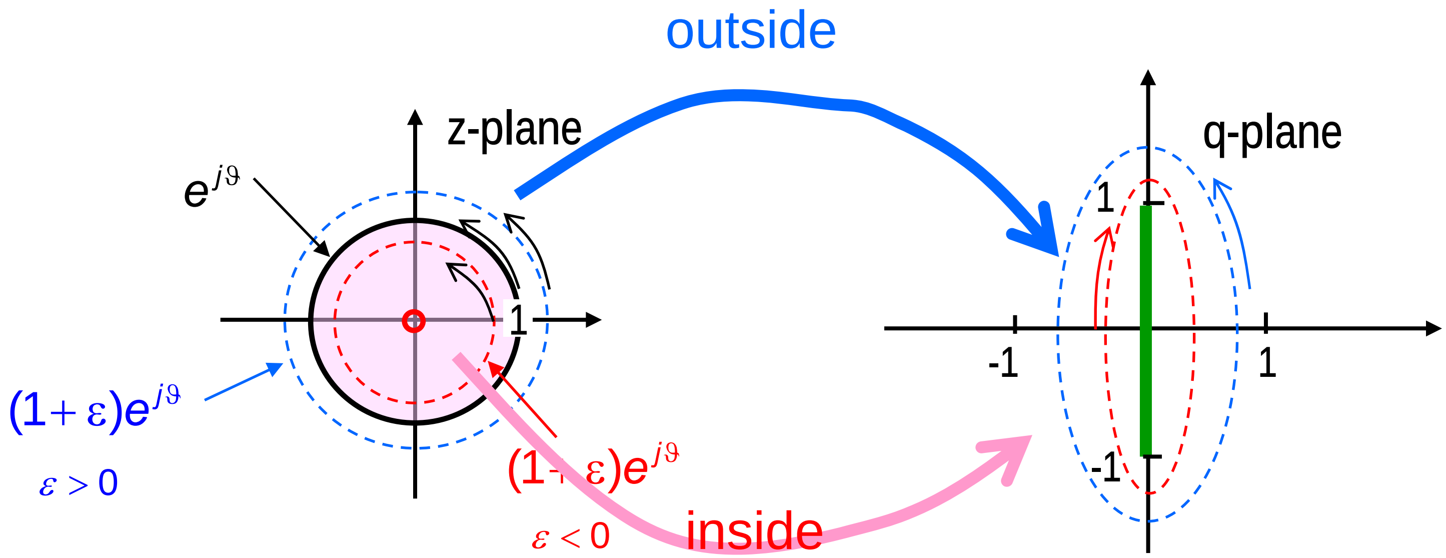
$$z^2 = 1 + 2qz$$

Poles: $\rho_1 \cdot \rho_2 = -1$

- **For any point outside of the interval *jsinθ* in the q-plane, there exist two curves passing that point, one is mapped from a circle |z| > 1, the other from a circuit |z| < 1.**

Both inside & outsize of |z| = 1 mapped to the region outside of the interval line.

Interpretation - 2



Both inside and outside of the unit circle are mapped to the region outside of the interval [$j\sin\theta$].